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Clebsch–Gordan series as a complete set of commuting operators

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Abstract. Matrices representing the Clebsch–Gordan series for the irreducible representations of a finite group form a complete set of commuting operators whose eigenvalues are the group characters and whose eigenvectors are the columns in the character table. These operators are dual to another complete set of commuting operators which represent the class multiplication structure constants. The duality between these two complete sets of commuting operators is made explicit.

1. Introduction

In quantum mechanics it is often useful to choose basis states which are eigenstates of a complete set of commuting observables. It is possible to formulate the representation theory of finite groups in this spirit. This was in fact proposed many years ago [1, 2]. More recently, this programme was stated in the language of quantum mechanics [3]. In this approach [4, 5] the complete set of commuting operators is the set of 'structure constants' for a finite group which are obtained from class multiplication.

In view of the usual duality which exists between classes and irreducible representations, one might expect that this approach dualizes. This is in fact so. It is possible to formulate the representation theory of finite groups from the 'structure constants' for multiplication of irreducible representations. The matrices which describe the Clebsch–Gordan series form a complete set of commuting operators whose eigenvalues form the group character table [6].

We exhibit the duality between the two approaches based on complete sets of commuting operators for class multiplication and representation multiplication. The complete sets of operators are constructed in section 2, their eigenvectors are computed in section 3, and the equivalence between them is established in section 4. The three descriptions of the representation theory of finite groups based on the character table and the dual approaches based on complete sets of commuting operators for classes and representations are equivalent. In section 5 we discuss the minimal information required to formulate these three approaches.

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2. Dual complete sets of commuting operators

We assume a finite group of order g has a complete set of n unitary irreducible representations Γ^i of dimension d_i, $i = 1, 2, \ldots, n$. The n classes C_α have order $n_\alpha, \alpha = 1, 2, \ldots, n$. The characters are denoted $\chi^i(\alpha)$. The character table is an $n \times n$ matrix χ with matrix elements $(\chi)^i_\alpha = \chi^i(\alpha)$ and with inverse χ^{-1} with matrix elements $(\chi^{-1})^\alpha_i = n_\alpha \chi^{i*}(\alpha)/g$. For convenience we state the orthogonality and completeness relations

$$
\begin{aligned}
(\chi\chi^{-1})^i_j &= \sum_\alpha \chi^i_\alpha (\chi^{-1})^\alpha_j = \sum_\alpha \chi^i(\alpha) \frac{n_\alpha}{g} \chi^{j^*}(\alpha) = \delta^i_j \\
(\chi^{-1}\chi)^\alpha_\beta &= \sum_i (\chi^{-1})^\alpha_i \chi^i_\beta = \sum_i \frac{n_\alpha}{g} \chi^{i^*}(\alpha)\chi^i(\beta) = \delta^\alpha_\beta .
\end{aligned}
\tag{1}
$$

All results are illustrated for the symmetric group S_3. The group has three irreducible representations: the identity representation Γ^3; the two-dimensional representation $\Gamma^{2,1}$; and the antisymmetric representation Γ^{1^3}. Dually, it has three classes, C_{1^3} which consists of the identity element, $C_{2,1}$ which contains the three transpositions and C_3 which contains the two three-cycles. The properties of S_3 are summarized in table 1.

2.1. *Representation multiplication and Clebsch–Gordan series*

The Clebsch–Gordan series is defined by

$$\Gamma^i \otimes \Gamma^j = \sum_k G^{ij}_k \Gamma^k = \Gamma^j \otimes \Gamma^i . \tag{2}$$

The coefficients G^{ij}_k are non-negative integers. These integers can be computed if the group character table is known. A simple character analysis on the compound character for the direct product representation $\Gamma^i \otimes \Gamma^j$ leads to

$$G^{ij}_k = \sum_\alpha \chi^i(\alpha)\chi^j(\alpha) \frac{n_\alpha}{g} \chi^{k*}(\alpha). \tag{3}$$

It is useful to collect the integers G^{ij}_k into n matrices, one for each representation Γ^i. This is conveniently done by introducing an n-dimensional linear vector space V_n with dual basis vectors $\langle\Gamma^j|, |\Gamma_k\rangle, \langle\Gamma^j|\Gamma_k\rangle = \delta^j_k$, on which Γ^i acts as follows

$$
\begin{aligned}
\langle\Gamma^j|\Gamma^i &= \langle\Gamma^j \otimes \Gamma^i| = \sum_k \langle\Gamma^j|\Gamma^i|\Gamma_k\rangle\langle\Gamma_k| \\
\{G(i)\}^j_k &= \langle\Gamma^j|\Gamma^i|\Gamma_k\rangle \equiv G^{ij}_k .
\end{aligned}
\tag{4}
$$

The matrix $G(i)$ defines the Clebsch–Gordan series generated by Γ^i. Since the Clebsch–Gordan series commutes (cf equation (2)) the n matrices $G(i)$ are mutually commuting. As a result they are simultaneously diagonalizable.

2.2. *Class multiplication and group structure constants*

A dual set of n mutually commuting operators may be constructed from the structure constants for class multiplication

$$\mathcal{C}_\alpha \otimes \mathcal{C}_\beta = \sum_\gamma C^\gamma_{\alpha\beta}\mathcal{C}_\gamma = \mathcal{C}_\beta \otimes \mathcal{C}_\alpha. \tag{2'}$$

The coefficients $C^\gamma_{\alpha\beta}$ are non-negative integers. These integers can be computed if the group character table is known. A simple 'class' analysis on the character for the product of two classes $\mathcal{C}_\alpha \otimes \mathcal{C}_\beta$ leads to

$$C^\gamma_{\alpha\beta} = \sum_i \frac{n_\beta}{g}\chi^i(\beta)\frac{n_\alpha}{d_i}\chi^i(\alpha)\chi^{i*}(\gamma) = C^{\gamma*}_{\alpha\beta}. \tag{3'}$$

It is useful to collect the integers $C^\gamma_{\alpha\beta}$ into n matrices, one for each class $\mathcal{C}_\alpha$. This is conveniently done by introducing another set of dual basis vectors in V_n, $\langle \mathcal{C}^\gamma|, |\mathcal{C}_\beta\rangle$, $\langle \mathcal{C}^\gamma|\mathcal{C}_\beta\rangle = \delta^\gamma_\beta$, on which $\mathcal{C}_\alpha$ acts as follows

$$\begin{aligned} \mathcal{C}_\alpha|\mathcal{C}_\beta\rangle &= |\mathcal{C}_\alpha \otimes \mathcal{C}_\beta\rangle = \sum_\gamma |\mathcal{C}_\gamma\rangle\langle \mathcal{C}^\gamma|\mathcal{C}_\alpha|\mathcal{C}_\beta\rangle \\ \{C(\alpha)\}^\gamma_\beta &= \langle \mathcal{C}^\gamma|\mathcal{C}_\alpha|\mathcal{C}_\beta\rangle \boxminus C^\gamma_{\alpha\beta}. \end{aligned} \tag{4'}$$

The matrix $C(\alpha)$ defines the class multiplication structure constants generated by $\mathcal{C}_\alpha$. Since class multiplication is commutative (cf equation (2')) the n matrices $C(\alpha)$ are mutually commuting. As a result they are also simultaneously diagonalizable.

3. Simultaneous eigenvectors

3.1. *Clebsch–Gordan series*

The eigenvalues of the matrices $G(i)$ are the elements in the ith row of the character table, $\chi^i(\mathcal{C}_1), \ldots, \chi^i(\mathcal{C}_n)$. The right eigenvector $|v(\beta)\rangle$ of each $G(i)$, when properly normalized, is the βth column of the character table χ. The right eigenvectors of $G(i)$ are

$$|v(\beta)\rangle = \sum_k |\Gamma^k\rangle\chi^k(\beta) \tag{5}$$

when normalized so that the first component is $+1$ ($\chi^k(\mathrm{Id}) = 1$). Then

$$G(i)|v(\beta)\rangle = \chi^i(\beta)|v(\beta)\rangle. \tag{6}$$

The left eigenvectors $\langle u(\beta)|$ are the rows of the inverse matrix χ^{-1}

$$\langle u(\beta)| = \sum_j \frac{n_\beta}{g}\chi^{j*}(\beta)\langle \Gamma_j|. \tag{7}$$

The matrices $G(i)$ are simultaneously diagonalized by the similarity transformation $\chi^{-1}G(i)\chi$:

$$\chi^{-1}G(i)\chi = \mathrm{diag}(\chi^i(\mathcal{C}_1), \chi^i(\mathcal{C}_2), \ldots, \chi^i(\mathcal{C}_n)). \tag{8}$$

The simultaneously commuting operators $G(3)$, $G(2,1)$, $G(1^3)$ are shown for S_3 in table 1. This table also shows the matrix of eigenvalues $E_\beta(i) = \chi^i(\beta)$ of the $G(i)$, the right eigenvectors $|v(\beta)\rangle$ of $G(i)$, and the diagonalized matrices $\chi^{-1}G(i)\chi$.

Table 1. The permutation group S_3 has three representations $= \Gamma, \Gamma, \Gamma$ and three classes $C(), C(), C()$. (*a*) Eigenvalue matrices $E_\alpha(G(i))$ for Clebsch–Gordan series and $E^i(C(\alpha))$ for class structure constants, labelled by operators and eigenvectors. (*b*) Eigenvector matrices χ contains right eigenvectors (columns) $|v(\beta)\rangle$ of $G(i)$ and left eigenvectors (rows) $\langle u'(i)|$ of $C(\alpha)$. χ^{-1} contains left eigenvectors (rows) $\langle u(\beta)|$ of $G(i)$ and right eigenvectors (columns) $|v'(i)\rangle$ of $C(\alpha)$. (*c*) Three mutually commuting operators $G(i)$ for the Clebsch–Gordan series and their diagonal form under similarity transformation $\chi^{-1}G(i)\chi$. Three mutually commuting operators $C(\alpha)$ for the class structure constants and their diagonal form under similarity transformation $\chi C(\alpha)\chi^{-1}$. (*d*) Expansion of $C(\alpha)$ in terms of $G(i)$ and expansion of $G(i)$ in terms of $C(\alpha)$.

(a) Eigenvalue Matrices

$E_\alpha(G(i))$

CG Series i \ Eigen Vector α	$\vert v([111])\rangle$ [111]	$\vert v([21])\rangle$ [21]	$\vert v([3])\rangle$ [3]
G([3]) [3]	1	1	1
G([21]) [21]	2	0	-1
G([111]) [111]	1	-1	1

$E^i(C(\alpha))$

Eigen Vector i \ Class Operator α	C([111]) [111]	C([21]) [21]	C([3]) [3]
$\langle u'([3])\vert$ [3]	1	3	2
$\langle u'([21])\vert$ [21]	1	0	-1
$\langle u'([111])\vert$ [111]	1	-3	2

$$\frac{n_\alpha}{d_i} E_\alpha(G(i)) = E^i(C(\alpha))$$

(b) Eigenvector Matrices χ

i \ α	$\vert v([111])\rangle$ [111]	$\vert v([21])\rangle$ [21]	$\vert v([3])\rangle$ [3]
$\langle u'([3])\vert$ [3]	1	1	1
$\langle u'([21])\vert$ [21]	2	0	-1
$\langle u'([111])\vert$ [111]	1	-1	1

χ^{-1}

i \ α	$\vert v'([3])\rangle$ [3]	$\vert v'([21])\rangle$ [21]	$\vert v'([111])\rangle$ [111]
$\langle u([111])\vert$ [111]	1/6	2/6	1/6
$\langle u([21])\vert$ [21]	3/6	0	-3/6
$\langle u([3])\vert$ [3]	2/6	-2/6	2/6

3.2. Class structure constants

The eigenvalues of the complete set of commuting operators $C(\alpha)$ are $n_\alpha \chi^{i*}(\alpha)/d_i$. The left eigenvectors $\langle u'(j)|$ are the rows of the character table χ

$$\langle u'(j)| = \sum_\beta \chi^j(\beta)\langle C^\beta| \tag{5'}$$

$$\langle u'(j)|C(\alpha) = \langle u'(j)|\frac{n_\alpha}{d_j}\chi^{j*}(\alpha). \tag{6'}$$

The right eigenvectors $|v'(k)\rangle$ of $C(\alpha)$ are the columns of the inverse matrix χ^{-1}

$$|v'(k)\rangle = \sum_\beta |C_\beta\rangle \frac{n_\beta}{g}\chi^{k*}(\beta). \tag{7'}$$

Table 1. (Continued).

(c) Complete Sets of Commuting Operators i	$G(i)$	$\chi^{-1}G(i)\chi$	$\chi C(\alpha)\chi^{-1}$	$C(\alpha)$	α
(3)	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	(1^3)
$(2,1)$	$\begin{pmatrix}0&1&0\\1&1&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}2&0&0\\0&0&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}3&0&0\\0&0&0\\0&0&-3\end{pmatrix}$	$\begin{pmatrix}0&1&0\\3&0&3\\0&2&0\end{pmatrix}$	$(2,1)$
(1^3)	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}2&0&0\\0&-1&0\\0&0&2\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&2&0\\2&0&1\end{pmatrix}$	(3)

(d) Expansions of Complete Sets of Commuting Operators	$C(\alpha)=\sum_i A_{\alpha i}\chi^{-2}G(i)\chi^{2}$	$G(i)=\sum_\alpha B^{i\alpha}\chi^{2}C(\alpha)\chi^{-2}$
	$A_{\alpha i}=\begin{array}{c} \alpha\backslash i \\ \begin{bmatrix}1&0&0\\-1/2&2&-1/2\\1/2&0&3/2\end{bmatrix}\end{array}$	$B^{i\alpha}=\begin{array}{c} i\backslash \alpha \\ \begin{bmatrix}1&0&0\\1/6&3/6&1/6\\-1/3&0&2/3\end{bmatrix}\end{array}$

The matrices $C(\alpha)$ are simultaneously diagonalized by the inverse similarity transformation $\chi C(\alpha)\chi^{-1}$:

$$\chi C(\alpha)\chi^{-1}=\mathrm{diag}\left(\frac{n_\alpha}{d_1}\chi^{1*}(\alpha),\frac{n_\alpha}{d_2}\chi^{2*}(\alpha),\ldots,\frac{n_\alpha}{d_n}\chi^{n*}(\alpha)\right). \quad (8')$$

The simultaneously commuting operators $C(1^3), C(2,1), C(3)$ are shown for S_3 in table 1. This table also shows the matrix of eigenvalues $E^i(\beta)=n_\beta\chi^{i*}(\beta)/d_i$ of the $C(\beta)$, their left and right eigenvectors, and the diagonalized matrices $\chi C(\alpha)\chi^{-1}$.

4. Transformations

Since the columns of the matrix χ are right eigenvectors of the matrices $G(i)$ and its rows are left eigenvectors of the matrices $C(\alpha)$, we have, combining equations (5) and (5′)

$$\chi=\sum_j\sum_\beta|\Gamma_j\rangle\chi^i(\beta)\langle\mathcal{C}^\beta|=\sum_j\sum_\beta|\Gamma_j\rangle\langle\Gamma^j|\mathcal{C}_\beta\rangle\langle\mathcal{C}^\beta|. \quad (9)$$

Similarly, combining equations (6) and (6′) or inverting equation (9) using equation (1) we find

$$\begin{aligned}|\chi^{-1}&=\sum_\alpha\sum_i|\mathcal{C}_\alpha\rangle\frac{n_\alpha}{g}\chi^{i*}(\alpha)\langle\Gamma^i|\\&=\sum_\alpha\sum_i|\mathcal{C}_\alpha\rangle\langle\mathcal{C}^\alpha|\Gamma_i\rangle\langle\Gamma^i|.\end{aligned} \quad (9')$$

That is, the character table is the non-unitary similarity transformation between the class basis and the representation basis in V_n.

Using this change of basis transformation we may write the complete sets of operators $G(i), C(\alpha)$ in their dual basis sets:

$$\begin{aligned} \chi^{-1}G(i)\chi| &= \sum_{\alpha,\beta} |\mathcal{C}_\alpha\rangle\langle\mathcal{C}^\alpha|G(i)|\mathcal{C}_\beta\rangle\langle\mathcal{C}^\beta| \\ \langle\mathcal{C}^\alpha|G(i)|\mathcal{C}_\beta\rangle &= \chi^i(\alpha)\delta^\alpha_\beta \end{aligned} \tag{10}$$

and

$$\begin{aligned} \chi C(\alpha)\chi^{-1} &= \sum_{i,j} |\Gamma_i\rangle\langle\Gamma^i|C(\alpha)|\Gamma_j\rangle\langle\Gamma^j| \\ \langle\Gamma^i|C(\alpha)|\Gamma_j\rangle &= \frac{n_\alpha}{d_i}\chi^{i*}(\alpha)\delta^i_j. \end{aligned} \tag{10$'$}$$

Equations (10) and (10′) exhibit the duality between the complete sets of commuting operators for representations and classes most succinctly. The commuting representation operators $G(i)$ are expressed in terms of their eigenvalues multiplied by class operators $|\mathcal{C}_\alpha\rangle\langle\mathcal{C}^\alpha|$, and conversely. Projection operators in the class basis and the representation basis are easily constructed as superpositions of diagonalized representation matrices $\chi^{-1}G(i)\chi$ and class matrices $\chi C(\alpha)\chi^{-1}$, respectively,

$$|\mathcal{C}_\alpha\rangle\langle\mathcal{C}^\alpha| = \sum_i \frac{n_\alpha}{g}\chi^{-1}G(i)\chi \tag{11}$$

$$|\Gamma_i\rangle\langle\Gamma^i| = \sum_\alpha \frac{d_i}{g}\chi C(\alpha)\chi^{-1}. \tag{11$'$}$$

Projectors in either basis are simply expressed in terms of projectors in the other basis

$$|\mathcal{C}_\alpha\rangle\langle\mathcal{C}^\alpha| = \sum_i |\Gamma_i\rangle\langle\Gamma^i|\delta_{i\alpha} \tag{12}$$

$$|\Gamma_i\rangle\langle\Gamma^i| = \sum_\alpha |\mathcal{C}_\alpha\rangle\langle\mathcal{C}^\alpha|\delta_{\alpha i}. \tag{12$'$}$$

These relations are used to express the $C(\alpha)$ as linear superpositions of the $G(i)$, and conversely. To express the $C(\alpha)$ in terms of the $G(i)$, we form superpositions of the projectors $|\Gamma_i\rangle\langle\Gamma^i|$ according to equation (10′) using equation (12′)

$$\begin{aligned} \chi C(\alpha)\chi^{-1}| &\overset{(10')}{=} \sum_j \frac{n_\alpha}{d_j}\chi^{j*}(\alpha)|\Gamma_j\rangle\langle\Gamma^j| \\ &\overset{(12')}{=} \sum_j \frac{n_\alpha}{d_j}\chi^{j*}(\alpha)\sum_\beta |\mathcal{C}_\beta\rangle\langle\mathcal{C}^\beta|\delta_{j\beta} \\ &\overset{(11')}{=} \sum_j\sum_\beta \frac{n_\alpha}{d_j}\chi^{j*}(\alpha)\delta_{j\beta}\sum_k \frac{n_\beta}{g}\chi^{k*}(\beta)\chi^{-1}G(k)\chi \end{aligned} \tag{13}$$

$$\chi C(\alpha)\chi^{-1} = \sum_k A_{\alpha k}\chi^{-1}G(k)\chi \tag{14}$$

$$A_{\alpha k} = \sum_\mu \frac{n_\alpha}{d_\mu}\chi^{\mu*}(\alpha)|\frac{n_\mu}{g}\chi^{k*}(\mu). \tag{15}$$

A dual computation involving equations (10), (12) and (11′) leads directly to

$$\chi^{-1}G(i)\chi = \sum_{\beta} B^{i\beta}\chi C(\beta)\chi^{-1} \tag{14′}$$

$$B^{i\beta} = \sum_{\mu} \chi^{i}(\mu)\frac{d_{\mu}}{g}\chi^{\mu}(\beta). \tag{15′}$$

It is easily verified that the matrices A and B are inverses:

$$\sum_{k} A_{\alpha k}B^{k\beta} = \delta_{\alpha}^{\beta} \qquad \sum_{\beta} B^{i\beta}A_{\beta j} = \delta_{j}^{i}. \tag{16}$$

The net result of this computation is to show that the class multiplication structure constants $C(\alpha)$ can be expressed directly in terms of the Clebsch–Gordan series $G(i)$ and conversely

$$C(\alpha) = \sum_{j} A_{\alpha j}\chi^{-2}G(j)\chi^{2} \tag{17}$$

$$G(i) = \sum_{\beta} B^{i\beta}\chi^{2}C(\beta)\chi^{-2}. \tag{17′}$$

5. Relations

Three approaches to the study of representations of finite groups have been described. These are the traditional approach based on the character table and the dual approaches of computing the eigenvalues and simultaneous eigenvectors for a complete set of commuting operators for representation multiplication (Clebsch–Gordan series) and class multiplication (class structure constants). These three approaches are equivalent. Any may be used to construct the other two. If the character table is known the dimensions of the representations are the integers in the column corresponding to the identity class, which is the only column whose entries are all positive integers. The group order is $g = \sum d_i^2$, and the number of elements in each class, n_α, is the positive integer in the column of $g\chi^{-1}$ corresponding to the identity representation. This is the only column of $g\chi^{-1}$ whose entries are all positive integers. The complete set of commuting operators for the Clebsch–Gordan series is constructed from equation (3) and the complete set of commuting operators for the class structure constants is constructed from equation (3′).

If the Clebsch–Gordan series is known, the character table is the matrix of eigenvalues of the simultaneous eigenvectors of the complete set of commuting operators. The dimensions of the representations and orders of the classes are determined as above from χ and χ^{-1}. The transformation matrix $A_{\alpha i}$ of equation (15) is constructed from this information and the complete set of simultaneously commuting class operators $C(\alpha)$ is constructed from equation (17).

If the class multiplication structure constants are known, the orders n_α and dimensions d_i are determined from the appropriate positive row and column of the

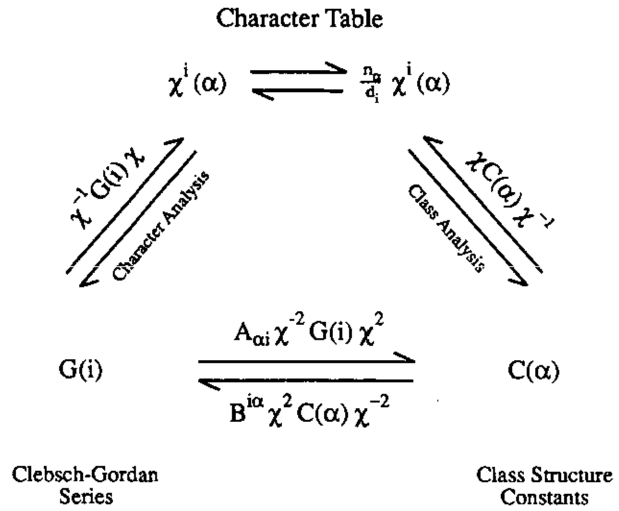

Figure 1. Three approaches to representation theory of finite groups based on the character table and two dual complete sets of commuting operators are equivalent.

eigenvalue matrix $E^i(C(\alpha)) = n_\alpha \chi^{i*}(\alpha)/d_i$. The character table is then easily reconstructed. The transformation matrix $B^{i\alpha}$ of equation (15′) is constructed from this information and the complete set of simultaneous operators $G(i)$ for the Clebsch–Gordan series is constructed from equation (17′).

The relations between these three approaches is summarized in figure 1. The traditional approach based on the group character table is the most efficient, being an order n^2 study (n^2 = number of matrix elements required). The dual approaches based on complete sets of commuting operators are order n^3 studies. Of the two, the approach based on the Clebsch–Gordan series is more useful if selection rules are known but the group is not. The dual approach is more convenient in the dual case. The two approaches reduce to order n^2 studies if one operator in the complete set has non-degenerate eigenvalues.

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